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Error Bounds for the Laplace Approximation for Definite Integrals

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Explicit error bounds are obtained for the well-known asymptotic expansion of integrals of the form

$$\int_a^b e^{-\lambda p(x)} q(x)\, dx,$$

in which λ is a large positive parameter, $p(x)$ and $q(x)$ are real differentiable functions, and $p'(x)$ has a simple zero in the finite or infinite range $[a,b]$. The bounds are expressed in terms of the supremum of a certain function, taken over $[a,b]$, and are asymptotic to the absolute value of the first neglected term in the expansion, as $\lambda \to \infty$. Several illustrative examples are given, including modified Bessel functions and the gamma function.

1. INTRODUCTION

Consider the integral

$$I = \int_a^b e^{-p(x)} q(x)\, dx, \tag{1.1}$$

in which the range of integration is real and may be finite or infinite. Assume that (i) $p(x)$ is real, attains its minimum value at an interior point ξ, say, of $[a,b]$ and increases fairly rapidly as $|x - \xi|$ increases; (ii) $q(x)$ is a relatively slowly-varying function. Then the underlying idea of the Laplace approximation is that almost the whole contribution of the integrand to I arises from the immediate neighborhood of ξ. In consequence, $p(x)$ and $q(x)$ may be replaced by the leading terms in their Taylor-series expansions at $x = \xi$, and the integration limits extended (if necessary) to $-\infty$ and $+\infty$. For example, in the case when $p'(\xi) = 0$, $p''(\xi) \neq 0$, and $q(\xi) \neq 0$, we obtain

$$I \doteqdot \int_a^b \exp\{-p(\xi) - \tfrac{1}{2}(x-\xi)^2 p''(\xi)\}\, q(\xi)\, dx$$

$$\doteqdot \int_{-\infty}^{\infty} \exp\{-p(\xi) - \tfrac{1}{2}(x-\xi)^2 p''(\xi)\}\, q(\xi)\, dx = e^{-p(\xi)} q(\xi) \left(\frac{2\pi}{p''(\xi)}\right)^{1/2}. \tag{1.2}$$

This procedure is particularly valuable when the integral contains a parameter: the approximation often possesses an asymptotic property with respect to the parameter. In particular, if $p(x)$ is replaced by $\lambda p(x)$, where λ is positive, then precise circumstances in which

$$\int_a^b e^{-\lambda p(x)} q(x)\, dx \sim e^{-\lambda p(\xi)} q(\xi) \left(\frac{2\pi}{\lambda p''(\xi)} \right)^{1/2} \tag{1.3}$$

as $\lambda \to \infty$ are well known; see, for example, [1]–[5].

The object of this paper is to supply explicit bounds for the error in the Laplace and related approximations, from which (i) asymptotic properties with respect to parameters are directly deducible, (ii) realistic numerical values can be determined. For earlier work on this problem see [6].

2. Formulation

By subdividing the range of integration and changing variables, we can reduce the problem of Section 1 to the approximation of an integral of the form

$$I(\lambda) = \int_0^b e^{-\lambda p(x)} q(x)\, dx, \tag{2.1}$$

in which $p(0) = p'(0) = 0$, and

$$p'(x) > 0 \qquad (0 < x \leqslant b). \tag{2.2}$$

The functions $p(x)$ and $q(x)$ may themselves depend on the positive parameter λ.

We assume that $p(x)$ and $q(x)$ can be expanded in series of ascending powers of x, given by

$$p(x) = p_2 x^2 + p_3 x^3 + p_4 x^4 + \ldots, \tag{2.3}$$

$$q(x) = q_0 + q_1 x + q_2 x^2 + \ldots. \tag{2.4}$$

For our purposes it suffices for these series to be asymptotic expansions in Poincaré's sense, as $x \to 0$. In particular, this includes the common case in which $p(x)$ and $q(x)$ are analytic at $x = 0$ and (2.3) and (2.4) are Maclaurin expansions. In consequence of (2.2), we have $p_2 \geqslant 0$. In this paper we exclude the case $p_2 = 0$; in other words, we restrict $x = 0$ to being a simple saddle-point of the integral (2.1).

Since $p(x)$ is an increasing function in the interval $(0, b)$, we may adopt $p = p(x)$ as a new integration variable. This gives

$$I(\lambda) = \int_0^{p(b)} e^{-\lambda p} f(p) \frac{dp}{p^{1/2}}, \tag{2.5}$$

where

$$f(p) = p^{1/2}(x) q(x) / p'(x). \tag{2.6}$$

Reversion of (2.3) yields an expansion of the form

$$x = c_1 p^{1/2} + c_2 p + c_3 p^{3/2} + \dots, \tag{2.7}$$

this series converging in a neighborhood of $p = 0$ if (2.2) converges, otherwise it is an asymptotic expansion for small p. Substitution of (2.7) in (2.6) yields

$$f(p) = a_0 + a_1 p^{1/2} + a_2 p + a_3 p^{3/2} + \dots, \tag{2.8}$$

the coefficients a_s being expressible in terms of p_s and q_s; in particular,

$$a_0 = q_0/(2p_2^{1/2}).$$

If, in (2.5), we replace the upper limit by infinity, substitute (2.8), and integrate formally term by term, we obtain the series

$$\sum_{s=0}^{\infty} \frac{a_s \Gamma\{(s+1)/2\}}{\lambda^{(s+1)/2}}. \tag{2.9}$$

The first term is one-half the Laplace approximation (1.3).[1] In the case when $p(x)$ and $q(x)$ are independent of λ, the whole series is an asymptotic expansion for $I(\lambda)$ as $\lambda \to \infty$; this is a consequence of Watson's well-known lemma [7], Section 8.3.

We now write

$$I(\lambda) = \sum_{s=0}^{n-1} \frac{a_s \Gamma\{(s+1)/2\}}{\lambda^{(s+1)/2}} + E_n(\lambda), \tag{2.10}$$

where n is an arbitrary positive integer at zero, and seek bounds for the error term $E_n(\lambda)$. In constructing these bounds we shall bear in mind the asymptotic value

$$E_n(\lambda) \sim \frac{a_n \Gamma\{(n+1)/2\}}{\lambda^{(n+1)/2}} + \frac{a_{n+1} \Gamma\{(n+2)/2\}}{\lambda^{(n+2)/2}} + \dots \qquad (\lambda \to \infty), \tag{2.11}$$

applicable when $p(x)$ and $q(x)$ are independent of λ.

We write

$$f(p) = \sum_{s=0}^{n-1} a_s p^{s/2} + f_n(p). \tag{2.12}$$

The remainder term $f_n(p)$ is a continuous function of p, with the property

$$f_n(p) \sim a_n p^{n/2} \qquad (p \to 0), \tag{2.13}$$

provided that $a_n \neq 0$. It is convenient to split $E_n(\lambda)$ in the form

$$E_n(\lambda) = E_n^{(2)}(\lambda) - E_n^{(1)}(\lambda), \tag{2.14}$$

[1] The other half is contributed by the integral over the range $a \leqslant x \leqslant 0$.

where

$$E_n^{(1)}(\lambda) = \sum_{s=0}^{n-1} a_s \int_{p(b)}^{\infty} e^{-\lambda p} p^{(s-1)/2}\, dp, \tag{2.15}$$

$$E_n^{(2)}(\lambda) = \int_0^{p(b)} e^{-\lambda p} f_n(p) \frac{dp}{p^{1/2}}, \tag{2.16}$$

and to consider the two parts separately in succeeding sections. If $p(b) = \infty$, then $E_n^{(1)}(\lambda) = 0$.

3. Bounds for $E_n^{(1)}(\lambda)$

Denoting $p(b)$ by β, and taking a new integration variable $\tau = p - \beta$, we may write

$$E_n^{(1)}(\lambda) = \sum_{s=0}^{n-1} a_s L_s\{p(b), \lambda\}, \tag{3.1}$$

where

$$L_s(\beta, \lambda) = \int_\beta^\infty e^{-\lambda p} p^{(s-1)/2}\, dp = e^{-\lambda\beta} \int_0^\infty e^{-\lambda\tau} (\tau + \beta)^{(s-1)/2}\, d\tau. \tag{3.2}$$

Clearly

$$L_0(\beta, \lambda) \leqslant \frac{e^{-\lambda\beta}}{\lambda\beta^{1/2}}. \tag{3.3}$$

For $s \geqslant 1$, we apply the inequality

$$\tau + \beta \leqslant \beta e^{\tau/\beta}.$$

This yields

$$L_s(\beta, \lambda) \leqslant \frac{\beta^{(s-1)/2} e^{-\lambda\beta}}{\lambda - \frac{1}{2}(s-1)\beta^{-1}}, \tag{3.4}$$

provided that $\lambda > \frac{1}{2}(s-1)\beta^{-1}$. When $s = 1$, the relation (3.4) is an equality. For other values of s and large values of λ, the satisfactory nature of the bound can be seen from the relation

$$L_s(\beta, \lambda) = \frac{\beta^{(s-1)/2} e^{-\lambda\beta}}{\lambda} \left\{1 + \frac{s-1}{2\beta\lambda} + O\left(\frac{1}{\lambda^2}\right)\right\} \qquad (\lambda \to \infty),$$

obtained by applying Watson's lemma to (3.2).

Substitution of (3.3) and (3.4) in (3.1) yields

$$|E_1^{(1)}(\lambda)| \leqslant \frac{e^{-\lambda p(b)} |a_0|}{\lambda p^{1/2}(b)}, \tag{3.5}$$

and

$$|E_n^{(1)}(\lambda)| \leqslant \frac{e^{-\lambda p(b)}}{\lambda - \{(\tfrac{1}{2}n-1)/p(b)\}} \sum_{s=0}^{n-1} |a_s| \{p(b)\}^{(s-1)/2} \qquad (n \geqslant 2), \tag{3.6}$$

provided that the denominator in the last relation is positive.

4. Bounds for $E_n^{(2)}(\lambda)$

To bound $E_n^{(2)}(\lambda)$, we majorize $f_n(p)$ and then replace the upper integration limit in (2.16) by infinity. One of the simplest results of this kind is given by

$$|E_n^{(2)}(\lambda)| \leqslant A_n \int_0^\infty e^{-\lambda p} p^{(n-1)/2}\, dp = \frac{A_n \Gamma\{(n+1)/2\}}{\lambda^{(n+1)/2}}, \tag{4.1}$$

where

$$A_n = \sup_{0 < p < p(b)} |p^{-n/2} f_n(p)|. \tag{4.2}$$

This form of majorant was used by Rosser [6]. However, A_n may not exist when $p(b) = \infty$, and in other cases it may be many times the size of $|a_n|$, causing (4.1) to overestimate grossly the actual error; compare (2.11).

Provided that $a_n \neq 0$, a more realistic bound is obtainable by using a majorant of the form

$$|f_n(p)| \leqslant |a_n| p^{n/2} e^{\phi_n \sqrt{p}}, \tag{4.3}$$

where ϕ_n is independent of p. The best value of ϕ_n is obviously

$$\phi_n = \sup_{0<p<p(b)} \left\{ \frac{1}{\sqrt{p}} \ln \left| \frac{f_n(p)}{a_n p^{n/2}} \right| \right\}. \tag{4.4}$$

Combination of (2.16) and (4.3) gives

$$|E_n^{(2)}(\lambda)| \leqslant |a_n| \int_0^\infty \exp(-\lambda p + \phi_n \sqrt{p})\, p^{(n-1)/2}\, dp. \tag{4.5}$$

If $\phi_n \leqslant 0$, we have immediately

$$|E_n^{(2)}(\lambda)| \leqslant \frac{|a_n| \Gamma\{(n+1)/2\}}{\lambda^{(n+1)/2}}. \tag{4.6}$$

Now suppose that $\phi_n > 0$. Taking a new integration variable $w = \sqrt{p}$, we find that

$$\int_0^\infty \exp(-\lambda p + \phi_n \sqrt{p})\, p^{(n-1)/2}\, dp = 2 \exp\left(\frac{\phi_n^2}{4\lambda}\right) \int_0^\infty \exp\left\{-\lambda \left(w - \frac{\phi_n}{2\lambda}\right)^2\right\} w^n\, dw.$$

Clearly

$$\int_0^{\phi_n/(2\lambda)} \exp\left\{-\lambda\left(w - \frac{\phi_n}{2\lambda}\right)^2\right\} w^n\, dw \leqslant \int_0^{\phi_n/(2\lambda)} w^n\, dw = \frac{1}{n+1}\left(\frac{\phi_n}{2\lambda}\right)^{n+1},$$

and

$$\int_{\phi_n/(2\lambda)}^{\infty} \exp\left\{-\lambda\left(w-\frac{\phi_n}{2\lambda}\right)^2\right\} w^n\,dw = \int_0^{\infty} e^{-\lambda t^2}\left(t+\frac{\phi_n}{2\lambda}\right)^n dt$$

$$= \sum_{s=0}^{n} \binom{n}{s}\left(\frac{\phi_n}{2\lambda}\right)^s \frac{\Gamma\{(n-s+1)/2\}}{2\lambda^{(n-s+1)/2}}.$$

Hence

$$|E_n^{(2)}(\lambda)| \leqslant \frac{|a_n|\,\Gamma\{(n+1)/2\}}{\lambda^{(n+1)/2}} \exp\left(\frac{\phi_n^2}{4\lambda}\right) \sum_{s=0}^{n+1} \alpha_{n,s}\left(\frac{\phi_n}{2\lambda^{1/2}}\right)^s, \tag{4.7}$$

where[2]

$$\alpha_{n,s} = \binom{n}{s}\frac{\Gamma\{(n-s+1)/2\}}{\Gamma\{(n+1)/2\}} \quad (s \leqslant n); \qquad \alpha_{n,n+1} = \frac{1}{\Gamma\{(n+3)/2\}}. \tag{4.8}$$

LEMMA 1.

$$\frac{\Gamma(x)}{\Gamma(x+\frac{1}{2})} \leqslant \left(\frac{\pi}{2x}\right)^{1/2} \qquad (x \geqslant \tfrac{1}{2}). \tag{4.9}$$

Proof. Denoting the logarithmic derivative of $\Gamma(x)$ by $\psi(x)$ and writing $\chi(x) = x^{1/2}\Gamma(x)/\Gamma(x+\frac{1}{2})$, we have

$$\frac{\chi'(x)}{\chi(x)} = \frac{1}{2x} + \psi(x) - \psi(x+\tfrac{1}{2}) = \tfrac{1}{4}\{\psi(x+1) - 2\psi(x+\tfrac{1}{2}) + \psi(x)\} = \tfrac{1}{8}\psi''(\zeta),$$

where $\zeta \in (x, x+1)$. Since

$$\psi''(\zeta) = -\int_0^{\infty} \frac{t^2 e^{-\zeta t}}{1-e^{-t}}\,dt \qquad (\zeta > 0),$$

it follows that $\chi'(x)$ is negative. Hence $\chi(x)$ attains its greatest value in the interval $\frac{1}{2} \leqslant x < \infty$ at $x = \frac{1}{2}$. This agrees with (4.9).

Repeated use of the lemma establishes that

$$\alpha_{n,s} \leqslant (n\pi)^{s/2}/s! \qquad (s = 0, 1, \ldots, n+1, \quad n \geqslant 1),$$

and substitution of this result in (4.7) yields the desired bound

$$|E_n^{(2)}(\lambda)| \leqslant \frac{|a_n|\,\Gamma\{(n+1)/2\}}{\lambda^{(n+1)/2}} \exp\left(\frac{\phi_n^2}{4\lambda} + \frac{\phi_n}{2}\sqrt{\frac{n\pi}{\lambda}}\right) \qquad (\phi_n \geqslant 0, n \geqslant 1), \tag{4.10}$$

[2] Somewhat curiously, as $s \to n+1$, the limiting form of the first of these expressions equals the second expression.

where ϕ_n is defined by (4.4). In terms of the original functions,

$$\phi_n = \sup_{0<x<b} \left\{ \frac{1}{p^{1/2}(x)} \ln \left| \frac{F_n(x)}{a_n p^{n/2}(x)} \right| \right\}, \tag{4.11}$$

where

$$F_n(x) = \frac{p^{1/2}(x) q(x)}{p'(x)} - \sum_{s=0}^{n-1} a_s p^{s/2}(x). \tag{4.12}$$

When a_n vanishes, we may use the same procedure with a suitably chosen constant in place of $|a_n|$ on the right of (4.3). This case is illustrated in Example 5 below.

Comparing the bounds of this and the previous section, we observe that the bound for $|E_n^{(1)}(\lambda)|$ is exponentially smaller than the corresponding bound for $|E_n^{(2)}(\lambda)|$ when λ is large. We also observe that the bound for $|E_n^{(2)}(\lambda)|$ has the same asymptotic form as the modulus of the leading term in (2.11), provided that $a_n \neq 0$.

5. Alternative Bound for $E_n^{(2)}(\lambda)$ When $p(x)$ and $q(x)$ are Even

If the coefficients p_s and q_s of odd suffix vanish, then the a_s of odd suffix also vanish. In this event, in place of (4.3) we may use the majorant

$$|f_{2n}(p)| \leqslant |a_{2n}| p^n e^{\psi_{2n} p}, \tag{5.1}$$

where

$$\psi_{2n} = \sup_{0<p<p(b)} \left\{ \frac{1}{p} \ln \left| \frac{f_{2n}(p)}{a_{2n} p^n} \right| \right\}. \tag{5.2}$$

Substitution of this inequality in (2.16) gives immediately

$$|E_{2n}^{(2)}(\lambda)| \leqslant \frac{|a_{2n}| \Gamma(n+\frac{1}{2})}{(\lambda - \psi_{2n})^{n+(1/2)}} \qquad (\lambda > \psi_{2n}). \tag{5.3}$$

This result is applicable to integrals of the form

$$I(\lambda) = \int_a^b e^{-\lambda p(x)} q(x)\, dx \tag{5.4}$$

in the case when $p(x)$ has a minimum at an interior point of $[a,b]$. We may suppose, without loss of generality, that this minimum occurs at $x=0$, that $p(0) = p'(0) = 0$, and that $p'(x)$ has the same sign as x throughout $[a,b]$. By truncation of one end of the range of integration, if necessary, we can arrange that

$$p(a) = p(b). \tag{5.5}$$

The contribution to $I(\lambda)$ from the range $0 \leqslant x \leqslant b$ may be transformed, as in Section 2, into the form (2.5). For the range $a \leqslant x \leqslant 0$, we have

$$\int_a^0 e^{-\lambda p(x)} q(x)\, dx = \int_0^{p(a)} e^{-\lambda p} \hat{f}(p) \frac{dp}{p^{1/2}}, \tag{5.6}$$

where

$$\hat{f}(p) = -\frac{p^{1/2}(x) q(x)}{p'(x)}, \tag{5.7}$$

the value of $p^{1/2}(x)$, again, being nonnegative. It is easily verified that the expansion corresponding to (2.8) is given by

$$\hat{f}(p) = a_0 - a_1 p^{1/2} + a_2 p - a_3 p^{3/2} + \dots. \tag{5.8}$$

Using (5.5), we derive

$$I(\lambda) = \int_0^{p(b)} e^{-\lambda p} \{f(p) + \hat{f}(p)\} \frac{dp}{p^{1/2}}. \tag{5.9}$$

Since only integer powers of p occur in the expansion of $f(p) + \hat{f}(p)$, this integral has the desired form.

In terms of the original variables we find that

$$f(p) + \hat{f}(p) = p^{1/2}(x) \left\{ \frac{q(x)}{p'(x)} - \frac{q(\hat{x})}{p'(\hat{x})} \right\}, \tag{5.10}$$

where $x \in [0, b]$ and $\hat{x}$ is the point in the interval $[a, 0]$ such that

$$p(\hat{x}) = p(x). \tag{5.11}$$

Remark. Because of the simplicity of (5.3) compared with (4.10), it is tempting to try using a majorant of the form (5.1) even in the case when the a_s of odd suffix do not vanish. When $a_{2n+1}/a_{2n} \leqslant 0$ this is feasible, but not otherwise. This is because

$$f_{2n}(p) = a_{2n} p^n \{1 + (a_{2n+1}/a_{2n}) p^{1/2} + O(p)\} \qquad (p \to 0).$$

If $a_{2n+1}/a_{2n} > 0$, then regardless of the value of ψ_{2n}, the function $e^{\psi_{2n} p}$ cannot grow quickly enough with p to majorize the quantity in braces. To put this another way, $\psi_{2n} = \infty$ when $a_{2n+1}/a_{2n} > 0$.

6. Evaluation of ϕ_n and ψ_{2n}

The principal difficulty in evaluating the bounds derived above is the determination of the number ϕ_n defined by (4.4) or (4.11), or, alternatively, the number ψ_{2n} defined by (5.2). The required supremum can always be found by numerical computation, but in every example that has been carried out, the supremum has been approached at one of the endpoints of the interval $0 < x < b$. In this section we discuss tests which can sometimes be used to establish this fact without recourse to computation. For convenience, we use the notation

$$w = p^{1/2}, \qquad g(w) = f(p), \qquad g_n(w) = f_n(p).$$

Application of Taylor's theorem to the expansion

$$g(w) = a_0 + a_1 w + a_2 w^2 + \ldots + a_{n-1} w^{n-1} + g_n(w) \tag{6.1}$$

gives

$$g_n(w) = w^n g^{(n)}(\eta)/n!, \tag{6.2}$$

where $0 < \eta < w$. Let $\hat{\phi}_n$ denote the greater of zero and

$$\sup_{0 < w < p^{1/2}(b)} \left\{ \frac{1}{w} \ln \left| \frac{g^{(n)}(w)}{g^{(n)}(0)} \right| \right\}, \tag{6.3}$$

so that

$$|g^{(n)}(w)| \leqslant |g^{(n)}(0)| \exp(\hat{\phi}_n w). \tag{6.4}$$

Then

$$|g_n(w)| \leqslant |a_n| w^n \exp(\hat{\phi}_n \eta) \leqslant |a_n| w^n \exp(\hat{\phi}_n w). \tag{6.5}$$

Comparison of this result with (4.3) shows that $\phi_n \leqslant \hat{\phi}_n$.

This result furnishes an alternative way of computing ϕ_n, or rather an upper bound for ϕ_n. Usually, however, the evaluation of $\hat{\phi}_n$ is as difficult as the evaluation of ϕ_n. The result is of some value, however, when $g(w)$ is a *completely monotonic* function[3] in the interval $0 < w < p^{1/2}(b)$. By this it is meant that each derivative of $g(w)$ is of constant sign, the sign alternating from one derivative to the next. Obviously, $|g^{(n)}(w)|$ then attains its maximum at $w = 0$; hence $\hat{\phi}_n = 0$ and (4.6) applies. It may be noticed that this particular result is equivalent to the simple error test of Steffensen applied to the integral

$$E_n^{(2)}(\lambda) = 2 \int_0^{p^{1/2}(b)} e^{-\lambda w^2} g_n(w)\, dw. \tag{6.6}$$

This test postulates that if consecutive remainder terms in a series expansion have opposite signs, then the truncation error is numerically less than the first neglected term in the series, and has the same sign [*9*].

In the general case, the essential problem in evaluating ϕ_n, ψ_{2n}, or $\hat{\phi}_n$, is the determination of a supremum of the form

$$\sup_{0<w<c} \left\{ \frac{\ln H(w)}{w} \right\},$$

[3] Also called an *alternating* function. Tests for complete monotonicity are given in [*1*] and [*8*].

where $(0, c)$ is a finite or infinite interval and $H(w)$ is a nonnegative function such that $H(0) = 1$. We have just considered the trivial situation in which $H'(w)$ is negative throughout $(0, c)$. Let us now suppose that $H'(w)$ is positive.

LEMMA 2. *Let $H(w)$ be a twice-differentiable function in the interval $(0, c)$ such that $H(0) = 1$ and $H'(w) \geqslant 0$.*

(i) *If $H'(w)/H(w)$ is nonincreasing, or equivalently, $H(w)H''(w) \leqslant H'^2(w)$, then $w^{-1}\ln H(w)$ is nonincreasing.*

(ii) *If $H'(w)/H(w)$ is nondecreasing, or equivalently, $H(w)H''(w) \geqslant H'^2(w)$, then $w^{-1}\ln H(w)$ is nondecreasing.*

This result is easily proved by differentiation. It can be applied to a power-series expansion in the following way.

LEMMA 3. *Let*

$$H(w) = h_0 + h_1 w + h_2 w^2 + \dots, \tag{6.7}$$

this expansion converging when $0 \leqslant w < c$. If $h_0 = 1$, $h_s \geqslant 0$ $(s \geqslant 1)$, and

$$(s+1)\, h_{s+1} h_{s-1} \geqslant s h_s^2 \qquad (s \geqslant 1), \tag{6.8}$$

then $w^{-1}\ln H(w)$ is nondecreasing.

The genesis of this result is that when (6.8) is an equality for all s, $H(w)$ is an exponential function and $w^{-1}\ln H(w)$ is a constant.

When $h_1 = 0$, the condition (6.8) implies that all the higher h_s vanish and the lemma is trivial. Now assume that $h_1 > 0$. We then have

$$H(w) = \sum_{s=0}^{\infty} \frac{\theta_1 \theta_2 \dots \theta_s}{s!} w^s,$$

where $0 < \theta_1 \leqslant \theta_2 \leqslant \theta_3 \leqslant \dots$. Hence

$$H(w)H''(w) - H'^2(w) = \sum_{s=0}^{\infty} \frac{k_s}{s!} w^s,$$

where

$$k_s = s! \sum_{r=0}^{s} \left\{ \frac{\theta_1 \theta_2 \dots \theta_{r+2}}{r!} \frac{\theta_1 \theta_2 \dots \theta_{s-r}}{(s-r)!} - \frac{\theta_1 \theta_2 \dots \theta_{r+1}}{r!} \frac{\theta_1 \theta_2 \dots \theta_{s-r+1}}{(s-r)!} \right\}$$

$$= \sum_{r=0}^{s} \binom{s}{r} \theta_1 \theta_2 \dots \theta_{r+1}\, \theta_1 \theta_2 \dots \theta_{s-r} (\theta_{r+2} - \theta_{s-r+1}).$$

Addition of this sum to its reverse gives

$$2k_s = \sum_{r=0}^{s} \binom{s}{r} \theta_1 \theta_2 \dots \theta_r \theta_1 \theta_2 \dots \theta_{s-r} \{\theta_{r+1}(\theta_{r+2} - \theta_{s-r+1}) + \theta_{s-r+1}(\theta_{s-r+2} - \theta_{r+1})\}.$$

The quantity in braces is not less than

$$\theta_{r+1}^2 - 2\theta_{r+1}\theta_{s-r+1} + \theta_{s-r+1}^2,$$

and is therefore nonnegative. Hence $k_s \geqslant 0$, from which the stated result follows.

7. Analytical Examples

Example 1[4]

$$I(\lambda) = \int_{-\infty}^{\infty} e^{-\lambda x^2} \ln(1 + x + x^2)\, dx \qquad (\lambda > 0). \tag{7.1}$$

Rewritten in the standard form of Section 2, this integral becomes

$$I(\lambda) = \int_0^{\infty} e^{-\lambda p} f(p) \frac{dp}{p^{1/2}}, \tag{7.2}$$

where $p = x^2$, and

$$f(p) = \tfrac{1}{2}\ln(1 + p^{1/2} + p) + \tfrac{1}{2}\ln(1 - p^{1/2} + p) = \tfrac{1}{2}\ln\left(\frac{1 - p^3}{1 - p}\right). \tag{7.3}$$

The expansion of $f(p)$ in ascending powers of p is given by (2.8), with $a_0 = 0$, $a_{2s+1} = 0$, and

$$a_{2s} = \frac{1}{2s} \quad (s \equiv 1 \text{ or } 2 \ (\mathrm{mod}\ 3)), \qquad a_{2s} = -\frac{1}{s} \quad (s \equiv 0 \ (\mathrm{mod}\ 3)), \qquad (s > 0). \tag{7.4}$$

From (2.10) and (2.14) we derive the desired expansion

$$I(\lambda) = \sum_{s=1}^{n-1} \frac{a_{2s}\Gamma(s + \frac{1}{2})}{\lambda^{s+(1/2)}} + E_{2n}^{(2)}(\lambda), \tag{7.5}$$

where $|E_{2n}^{(2)}(\lambda)|$ is bounded by (5.3). It remains to evaluate the quantity ψ_{2n} defined by (5.2) with $p(b) = \infty$.

In the present case

$$f_{2n}(p) = \tfrac{1}{2}\ln\left(\frac{1 - p^3}{1 - p}\right) - \sum_{s=1}^{n-1} a_{2s} p^s. \tag{7.6}$$

[4] [3], Section 4.1.

Hence, as $p \to \infty$, we have

$$\frac{1}{p} \ln \left| \frac{f_{2n}(p)}{a_{2n} p^n} \right| \to 0. \tag{7.7}$$

Therefore $\psi_{2n} \geqslant 0$. Also, as $p \to 0$,

$$\frac{1}{p} \ln \left| \frac{f_{2n}(p)}{a_{2n} p^n} \right| \to \frac{a_{2n+2}}{a_{2n}}. \tag{7.8}$$

Since the last quantity is positive when $n \equiv 1 \pmod 3$ and negative otherwise, it is reasonable to conjecture that

$$\psi_{2n} = \frac{a_{2n+2}}{a_{2n}} = \frac{n}{n+1} \quad (n \equiv 1 \pmod 3)), \qquad \psi_{2n} = 0 \quad (n \equiv 2 \text{ or } 0 \pmod 3)); \tag{7.9}$$

equivalently,

$$\left.\begin{aligned} |f_{6m+2}(p)| &\leqslant \frac{p^{3m+1}}{6m+2} \exp\left\{\frac{(3m+1)p}{3m+2}\right\}, \\ |f_{6m+4}(p)| &\leqslant \frac{p^{3m+2}}{6m+4}, \\ |f_{6m+6}(p)| &\leqslant \frac{p^{3m+3}}{3m+3}, \end{aligned}\right\} \tag{7.10}$$

for any nonnegative integer m. These inequalities may be established as follows.

Differentiation of (7.6) and expansion of the result yields the identity

$$f'_{2n}(p) = \frac{p^{n-1}\{n a_{2n}(1+p) + (n+1) a_{2n+2} p\}}{1 + p + p^2} \qquad (n \geqslant 1). \tag{7.11}$$

With the aid of (7.4), it is easy to verify that for positive p, the first derivatives of the right sides of the inequalities (7.10) exceed the absolute values of the expression (7.11) for $n = 3m+1$, $3m+2$, $3m+3$, respectively. Each of the inequalities (7.10) is satisfied at $p = 0$; their validity for all positive values of p is now obvious.

Summarizing the results of this example, we have shown that the remainder term $E_{2n}^{(2)}(\lambda)$ in the expansion (7.5) is bounded in absolute value by the first neglected term of the expansion if $n = 0$ or $2 \pmod 3$, and by

$$\{1 - n/(n\lambda + \lambda)\}^{-n-(1/2)}$$

times the first neglected term if $n \equiv 1 \pmod 3$, provided that in the latter event $\lambda > n/(n+1)$.

Example 2. The modified Bessel function I_0 with large argument
Consider the function

$$I_0(\lambda) = \frac{1}{\pi}\int_0^\pi e^{\lambda\cos x}\,dx. \tag{7.12}$$

The principal contribution of the integrand comes from the neighborhood of $x=0$. Owing to the presence of a second saddle-point at $x=\pi$, however, it is necessary to break the range of integration before our analysis is applicable. For simplicity we separate the range at its midpoint, but other subdivisions might lead to slightly better error bounds in certain circumstances.

Thus we write

$$I_0(\lambda) = \frac{e^\lambda}{\pi}\{I(\lambda) + E^{(3)}(\lambda)\}, \tag{7.13}$$

where

$$I(\lambda) = \int_0^{\pi/2} e^{-\lambda(1-\cos x)}\,dx, \qquad E^{(3)}(\lambda) = \int_{\pi/2}^{\pi} e^{-\lambda(1-\cos x)}\,dx. \tag{7.14}$$

If $\frac{1}{2}\pi \leqslant x \leqslant \pi$, then $1-\cos x \geqslant 2x/\pi$. Hence a bound for the second integral is given by

$$E^{(3)}(\lambda) \leqslant \int_{\pi/2}^{\pi} e^{-2\lambda x/\pi}\,dx < \frac{\pi}{2\lambda}e^{-\lambda}. \tag{7.15}$$

For $I(\lambda)$ we have, in the notation of Section 2, $p = 1-\cos x$ and

$$f(p) = \frac{1}{(2-p)^{1/2}} = \sum_{s=0}^{\infty} a_{2s}p^s \qquad (0 \leqslant p < 2), \tag{7.16}$$

where

$$a_0 = \frac{1}{\sqrt{2}}, \qquad a_{2s} = \frac{1.3.5\ldots(2s-1)}{4^s s!\sqrt{2}} \qquad (s \geqslant 1). \tag{7.17}$$

In the bounds (3.5) and (3.6), we have $p(b) = p(\frac{1}{2}\pi) = 1$. Hence

$$|E_1^{(1)}(\lambda)| \leqslant \frac{e^{-\lambda}}{\lambda\sqrt{2}}, \tag{7.18}$$

and

$$|E_{2n-1}^{(1)}(\lambda)| \leqslant \frac{e^{-\lambda}}{\lambda-n+\frac{3}{2}}\sum_{s=0}^{n-1} a_{2s} < \frac{e^{-\lambda}}{\lambda-n+\frac{3}{2}} \qquad (\lambda > n-\tfrac{3}{2},\ n \geqslant 2), \tag{7.19}$$

since

$$\sum_{s=0}^{\infty} a_{2s} = f(1) = 1.$$

Next, we consider the bound (5.3). The location of the supremum in (5.2) can be found in the present example by use of Lemma 3 of Section 6. In the notation of this lemma

$$w = p, \qquad H(p) = \frac{f_{2n}(p)}{a_{2n} p^n}, \qquad h_s = \frac{a_{2n+2s}}{a_{2n}}.$$

If $N \equiv n + s$ and $s \geqslant 1$, then from (7.17) we see that

$$\frac{(s+1) h_{s+1} h_{s-1}}{s h_s^2} = \frac{(s+1)(2N+1) N}{s(2N-1)(N+1)} = 1 + \frac{2N^2 + N + s}{(2N^2 + N - 1) s} > 1.$$

Therefore, the maximum value of $p^{-1} \ln H(p)$ is attained at $p = 1$. This gives

$$\psi_{2n} = \ln\left(\frac{1 - a_0 - a_2 - \ldots - a_{2n-2}}{a_{2n}}\right). \tag{7.20}$$

Numerical values computed from this formula are as follows:

$\psi_0 = 0.35, \quad \psi_2 = 0.50, \quad \psi_4 = 0.56, \quad \psi_6 = 0.59, \quad \psi_8 = 0.61, \quad \psi_{10} = 0.62.$

The aggregate of (7.15), (7.18), (7.19), (7.20), and (5.3) comprises the desired bound for the error term

$$\hat{E}_{2n}(\lambda) = E_{2n}^{(2)}(\lambda) - E_{2n-1}^{(1)}(\lambda) + E^{(3)}(\lambda) \tag{7.21}$$

in the expansion

$$I_0(\lambda) = \frac{e^\lambda}{\pi}\left\{\sum_{s=0}^{n-1} \frac{a_{2s}\Gamma(s+\frac{1}{2})}{\lambda^{s+(1/2)}} + \hat{E}_{2n}(\lambda)\right\}. \tag{7.22}$$

For example, taking $n = 1$ and bearing in mind that $E_1^{(1)}(\lambda)$ and $E^{(3)}(\lambda)$ are both positive, we obtain on combination

$$I_0(\lambda) = e^\lambda\left\{\frac{1}{(2\pi\lambda)^{1/2}} + \frac{\hat{E}_2(\lambda)}{\pi}\right\}, \tag{7.23}$$

where

$$\frac{|\hat{E}_2(\lambda)|}{\pi} \leqslant \frac{1}{(2\pi)^{1/2}} \frac{1}{8(\lambda - 0.50)^{3/2}} + \frac{e^{-\lambda}}{2\lambda} \qquad (\lambda > 0.50). \tag{7.24}$$

Other bounds for the remainder terms in the asymptotic expansions of modified Bessel functions for large arguments are included in more general results due to Meijer [*10*], [*11*], and the present writer [*12*], the former being derived from infinite contour integral representations and the latter from differential-equation theory. However, in the particular circumstances of the present example, that is, zero order and real argument, the bounds we have just derived appear to be the only ones whose ratios to the actual errors tend to unity when the argument λ tends to infinity.

Example 3. Modified Bessel function K_ν with large argument

Consider the integral

$$e^{\lambda} K_\nu(\lambda) = \int_0^\infty e^{-\lambda(\cosh x - 1)} \cosh \nu x \, dx. \tag{7.25}$$

In the notation of Section 2, we have $p(x) = \cosh x - 1$, $q(x) = \cosh \nu x$, and

$$f(p) = \frac{\cosh \nu x}{\sqrt{2} \cosh \frac{1}{2}x} = \frac{\{1 + p + (2p + p^2)^{1/2}\}^\nu + \{1 + p - (2p + p^2)^{1/2}\}^\nu}{2\sqrt{2}(1 + \frac{1}{2}p)^{1/2}}. \tag{7.26}$$

Thus

$$f(p) = \sum_{s=0}^{\infty} a_{2s} p^s \qquad (0 \leqslant p < 2), \tag{7.27}$$

where $a_0 = 1/\sqrt{2}$, and[5]

$$a_{2s} = \frac{(4\nu^2 - 1^2)(4\nu^2 - 3^2) \ldots \{4\nu^2 - (2s-1)^2\}}{(2s)!\, 2^{s+(1/2)}} \qquad (s = 1, 2, \ldots). \tag{7.28}$$

Application of the results of Sections 2 and 5 gives

$$e^{\lambda} K_\nu(\lambda) = \sum_{s=0}^{n-1} \frac{a_{2s} \Gamma(s + \frac{1}{2})}{\lambda^{s+(1/2)}} + E_{2n}^{(2)}(\lambda), \tag{7.29}$$

where $|E_{2n}^{(2)}(\lambda)|$ is bounded by (5.3). In the case $\nu = 0$, we have

$$f(p) = \frac{1}{(2 + p)^{1/2}}. \tag{7.30}$$

Clearly $|f^{(n)}(p)|$ attains its maximum value at $p = 0$. Hence by reasoning similar to that of the second paragraph of Section 6, we see that the ψ_{2n} of (5.2) is zero.

This result may be extended to integer values of ν by use of the relation

$$2 \cosh 2\nu t = \sum_{s=0}^{\nu} (-)^s \frac{2\nu}{2\nu - s} \binom{2\nu - s}{s} (2 \cosh t)^{2\nu - 2s} \qquad (\nu = 1, 2, 3, \ldots). \tag{7.31}$$

Setting $t = \frac{1}{2}x$, we derive

$$f(p) = \sum_{s=0}^{\nu} (-)^s \frac{2\nu}{2\nu - s} \binom{2\nu - s}{s} 2^{\nu - s - 1} (2 + p)^{\nu - s - (1/2)}. \tag{7.32}$$

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[5] Probably the easiest way to derive (7.28) is by way of the differential equation $(2p + p^2) f''(p) + (1 + 2p) f'(p) + (\frac{1}{4} - \nu^2) f(p) = 0$.

Repeated differentiation yields an expansion of the form

$$f^{(n)}(p) = \sum_{s=0}^{\nu} \frac{A_{n,s}}{(2+p)^{s+n-\nu+(1/2)}}. \tag{7.33}$$

If $n \geqslant \nu$, then it is easily verified that all the coefficients $A_{n,s}$ have the same sign. Therefore, in these circumstances, $|f^{(n)}(p)|$ attains its maximum at $p = 0$, and again $\psi_{2n} = 0$.

Accordingly, we have shown that when ν is zero or a positive integer, the error term $E_{2n}^{(2)}(\lambda)$ in the asymptotic expansion (7.29) is bounded in magnitude by the first neglected term, provided that $n \geqslant \nu$. This example is purely illustrative, however, because the result has been established in another way; indeed it suffices that ν be real and $n > |\nu| - \frac{1}{2}$ ([7], Section 7.4).

When n does not exceed $|\nu| - \frac{1}{2}$, values of ψ_{2n} may be computed from (5.2). Alternative bounds available from differential-equation theory [*12*] make no distinction between $n \gtrless |\nu| - \frac{1}{2}$.

8. Numerical Examples

Example 4. *The gamma function*

An integral representation for the gamma function of real positive argument λ is provided by

$$\Gamma(\lambda) = \frac{1}{\lambda} \int_0^\infty e^{-t}\, t^\lambda\, dt = \lambda^\lambda e^{-\lambda} \int_0^\infty (u\, e^{1-u})^\lambda\, du. \tag{8.1}$$

We shift the maximum value of the integrand from $u = 1$ to the origin by means of the substitution $u = 1 + x$. This gives

$$\Gamma(\lambda) = \lambda^\lambda e^{-\lambda} \int_{-1}^\infty e^{-\lambda p(x)}\, dx, \tag{8.2}$$

where

$$p(x) = x - \ln(1 + x). \tag{8.3}$$

We may now either subdivide the range of the new integral at $x = 0$ and treat the two parts separately by the theory of Section 4, or apply the theory of Section 5 directly to (8.2). For illustrative purposes let us pursue the latter course. We begin by noting that the preliminary condition (5.5) is satisfied.

Reversion of the expansion

$$p(x) = \frac{x^2}{2} - \frac{x^3}{3} + \frac{x^4}{4} - \cdots \qquad (|x| < 1) \tag{8.4}$$

yields,

$$x = \sqrt{2}p^{1/2} + \frac{2}{3}p + \frac{\sqrt{2}}{18}p^{3/2} - \frac{2}{135}p^2 + \frac{\sqrt{2}}{1080}p^{5/2} + \ldots, \tag{8.5}$$

the reverted expansion converging for all sufficiently small $|p|$. Therefore

$$f(p) = p^{1/2}\frac{dx}{dp} = \sum_{s=0}^{\infty} a_s p^{s/2}, \tag{8.6}$$

where

$$a_0 = \frac{\sqrt{2}}{2}, \qquad a_1 = \frac{2}{3}, \qquad a_2 = \frac{\sqrt{2}}{12}, \qquad a_3 = -\frac{4}{135}, \qquad a_4 = \frac{\sqrt{2}}{432}, \ldots \tag{8.7}$$

Since $p'(x) = x/(1+x)$, we obtain from (5.10) and (5.8)

$$f(p) + \hat{f}(p) = \left(\frac{1}{x} - \frac{1}{\hat{x}}\right) p^{1/2}(x) = 2\sum_{s=0}^{\infty} a_{2s} p^s, \tag{8.8}$$

where $\hat{x}$ and x are related by

$$p(\hat{x}) = p(x), \tag{8.9}$$

x being positive and $\hat{x}$ negative.

The desired expansion is therefore

$$\Gamma(\lambda) = \lambda^\lambda e^{-\lambda}\left\{\sum_{s=0}^{n-1} \frac{2a_{2s}\Gamma(s+\frac{1}{2})}{\lambda^{s+(1/2)}} + E_{2n}^{(2)}(\lambda)\right\}, \tag{8.10}$$

where

$$|E_{2n}^{(2)}(\lambda)| \leqslant \frac{2|a_{2n}|\,\Gamma(n+\frac{1}{2})}{(\lambda - \psi_{2n})^{n+(1/2)}} \qquad (\lambda > \psi_{2n}), \tag{8.11}$$

$$\psi_{2n} = \sup_{0<x<\infty}\left\{\frac{1}{p(x)} \ln\left|\frac{F_{2n}(x)}{2a_{2n}p^n(x)}\right|\right\}, \tag{8.12}$$

and

$$F_{2n}(x) = \left(\frac{1}{x} - \frac{1}{\hat{x}}\right) p^{1/2}(x) - 2\sum_{s=0}^{n-1} a_{2s} p^s(x). \tag{8.13}$$

Numerical values of $F_{2n}(x)$ have been computed for the range $0 \leqslant x < \infty$, the requisite values of $\hat{x}$ being found by solving (8.9) with the aid of Newton's rule. The calculations show that the supremum in (8.12) is approached as $x \to 0$ for $n = 0, 1, 3$, and as $x \to \infty$ for $n = 2$. To four decimal places, the values of the leading ψ_{2n} are given by

$$\psi_0 = 0.1667, \qquad \psi_2 = 0.0278, \qquad \psi_4 = 0.0000, \qquad \psi_6 = 0.0245.$$

This appears to be the first time that error bounds have been evaluated directly for the expansion (8.10). Nevertheless, the example is mainly illustrative. It is well known ([13], Section 79) that in the corresponding expansion for the logarithm, given by

$$\ln \Gamma(\lambda) - (\lambda - \tfrac{1}{2}) \ln \lambda + \lambda - \tfrac{1}{2} \ln (2\pi) \sim \sum_{s=0}^{\infty} \frac{B_{2s+2}}{(2s+1)(2s+2)\lambda^{2s+1}}, \quad (8.14)$$

in which B_{2s+2} denotes the Bernoulli number of order $2s+2$, the remainder term is numerically less than the first neglected term (and has the same sign), when $\lambda > 0$.

Example 5. Modified Bessel functions K_λ of large order

As our final example we consider the Debye asymptotic expansion for the modified Bessel function

$$K_\lambda(\lambda\rho) = \tfrac{1}{2} \int_{-\infty}^{\infty} e^{\lambda(t - \rho \cosh t)}\, dt, \quad (8.15)$$

for positive ρ and large positive λ. The integrand has a maximum at $t = \sinh^{-1}(1/\rho)$. This is shifted to the origin by the transformation $t = x + \sinh^{-1}(1/\rho)$. Thus we derive

$$K_\lambda(\lambda\rho) = \tfrac{1}{2} \exp\{\lambda \sinh^{-1}(1/\rho) - \lambda(1+\rho^2)^{1/2}\} I(\lambda), \quad (8.16)$$

where

$$I(\lambda) = \int_{-\infty}^{\infty} e^{-\lambda p(x)}\, dx, \quad (8.17)$$

and

$$p(x) = (1+\rho^2)^{1/2}(\cosh x - 1) + \sinh x - x. \quad (8.18)$$

Reversion of the Maclaurin expansion for $p(x)$ leads to

$$f(p) \equiv \frac{p^{1/2}(x)}{p'(x)} = \sum_{s=0}^{\infty} a_s p^{s/2}, \quad (8.19)$$

where

$$a_0 = \frac{\sqrt{2}}{2\sigma}, \qquad a_1 = -\frac{1}{3\sigma^4}, \qquad a_2 = -\frac{\sqrt{2}}{8\sigma^3} + \frac{5\sqrt{2}}{24\sigma^7}, \qquad a_3 = \frac{4}{15\sigma^6} - \frac{8}{27\sigma^{10}}, \ldots, \quad (8.20)$$

and $\sigma \equiv (1+\rho^2)^{1/4}$. Applying the theory of Section 4 to the integral (8.17), we find that

$$I(\lambda) = I_1(\lambda) + I_2(\lambda), \quad (8.21)$$

where

$$I_1(\lambda) = \int_0^{\infty} e^{-\lambda p(x)}\, dx = \sum_{s=0}^{n-1} \frac{a_s \Gamma\{(s+1)/2\}}{\lambda^{(s+1)/2}} + E_{n,1}^{(2)}(\lambda), \quad (8.22)$$

$$I_2(\lambda) = \int_0^\infty e^{-\lambda p(-x)}\,dx = \sum_{s=0}^{n-1} (-)^s \frac{a_s \Gamma\{(s+1)/2\}}{\lambda^{(s+1)/2}} + E_{n,2}^{(2)}(\lambda), \tag{8.23}$$

and $E_{n,1}^{(2)}(\lambda)$, $E_{n,2}^{(2)}(\lambda)$ are bounded by expressions of the form (4.10).

Let us evaluate these bounds in the case $n = 2$. Combination of (8.22) and (8.23) yields

$$I(\lambda) = \frac{1}{\sigma}\left(\frac{2\pi}{\lambda}\right)^{1/2} + E_2^{(2)}(\lambda), \tag{8.24}$$

where

$$|E_2^{(2)}(\lambda)| \leqslant \frac{2|a_2|\,\Gamma(3/2)}{\lambda^{3/2}} \exp\left\{\frac{\phi_2{}^2}{4\lambda} + \phi_2\left(\frac{\pi}{2\lambda}\right)^{1/2}\right\}, \tag{8.25}$$

$$\phi_2 = \sup_{-\infty < x < \infty} \left\{\frac{1}{p^{1/2}(x)} \ln\left|\frac{F_2(x)}{a_2 p(x)}\right|\right\}, \tag{8.26}$$

and

$$F_2(x) = \pm\frac{p^{1/2}(x)}{p'(x)} - \frac{\sqrt{2}}{2\sigma} \pm \frac{p^{1/2}(x)}{3\sigma^4}. \tag{8.27}$$

In the last equality the upper or lower signs are taken according as $x \gtrless 0$; $p^{1/2}(x)$ is positive in both cases.

In (8.25), the coefficient a_2 is given by (8.20). As a function of σ, it vanishes at $\sigma = (5/3)^{1/4}$, causing ϕ_2 to become infinite and the bound (8.25) to break down in the neighborhood of this value. This failure can be avoided, if we replace $|a_2|$ in the majorant (4.3) by τ_2/σ, where τ_2 is the supremum of $\sigma|a_2|$ in the range $1 \leqslant \sigma < \infty$.[6] It is easily verified that $\tau_2 = \sqrt{2}/12$; accordingly, we have

$$|E_2^{(2)}(\lambda)| \leqslant \frac{(2\pi)^{1/2}}{12\sigma\lambda^{3/2}} \exp\left\{\frac{\chi_2{}^2}{4\lambda} + \chi_2\left(\frac{\pi}{2\lambda}\right)^{1/2}\right\} \qquad (\lambda > 0), \tag{8.28}$$

where

$$\chi_2 = \sup\left\{\frac{1}{p^{1/2}(x)} \ln\left|\frac{12\sigma F_2(x)}{\sqrt{2}p(x)}\right|\right\}. \tag{8.29}$$

Furthermore, if this supremum is evaluated with respect to $1 \leqslant \sigma < \infty$ as well as $-\infty < x < \infty$, *then the bound* (8.28) *is uniform with respect to* $\rho \in [0, \infty]$. Numerical calculations yield the value $\chi_2 = 0.251$, this value being approached when $\sigma = 1$ and $x \to 0$ through negative values. Correspondingly,

$$\tfrac{1}{4}\chi_2{}^2 = 0.016, \qquad \chi_2(\pi/2)^{1/2} = 0.315.$$

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[6] Instead of τ_2/σ, the supremum of $|a_2|$ could be used, but this would have the disadvantage of causing the ratio of the error bound to the approximation to become infinite as $\sigma \to \infty$.

Error bounds for this example are again available from differential-equation theory [*14*]. In the present notation, this reference gives the uniform bound

$$|E_2^{(2)}(\lambda)| \leqslant \frac{(2\pi)^{1/2}k}{\sigma\lambda^{3/2}} \exp\left(\frac{k}{\lambda}\right) \qquad (\lambda > 0, \rho > 0), \tag{8.30}$$

where

$$k = \frac{1}{6\sqrt{5}} + \frac{1}{12} = 0.158\ldots.$$

As $\lambda \to \infty$, this bound exceeds the right-hand side of (8.28) by the factor $1 + (2/\sqrt{5}) = 1.89\ldots$.

9. Summary and Conclusions

In this paper we have considered the well-known asymptotic expansion for integrals of the form

$$\int_a^b e^{-\lambda p(x)} q(x)\, dx,$$

for large positive values of the parameter λ, in cases where $p(x)$ and $q(x)$ are real differentiable functions and $p'(x)$ has a simple zero in the finite or infinite range $[a, b]$. In the first part (Sections 1–5) we constructed explicit bounds for the error terms associated with the expansion. These bounds are expressed in terms of the supremum of a certain function with respect to the range of integration. The bounds have the desirable property of being asymptotic to the absolute value of the first neglected term in the expansion as $\lambda \to \infty$. Analytical tests for locating the required suprema were given in Section 6 and applied to examples in Section 7. In cases where the tests are inapplicable numerical computation can be used; this was illustrated by examples in Section 8.

The principal difficulty in evaluating the error bounds is the location of the suprema referred to in the preceding paragraph. In every example carried out so far, it has transpired that the suprema occur at the zero of $p'(x)$ or at one of the endpoints of the range of integration. This suggests that it may be worthwhile to attempt the development of further analytical tests for locating the suprema.

Compared with error bounds obtained by application of the asymptotic theory of ordinary differential equations (in cases where such a comparison is possible), the present bounds are slightly sharper but more difficult to evaluate on the whole. Perhaps the last observation is not altogether surprising: it is generally true that higher *terms* in an asymptotic expansion can be obtained more readily from a differential equation than from an integral representation.

Finally, the methods used in this paper appear to be capable of extension to integrals having saddle-points of higher order, that is, points at which $p'(x)$ has a multiple zero.

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